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Monotone Preferences over Information

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Juan Dubra and Federico Echenique

Abstract

We consider preference relations over information that are monotone: more information is preferred to less. We prove that, if a preference relation on information about an uncountable set of states of nature is monotone, then it is not representable by a utility function.

KEYWORDS: value of information, Blackwell´s Theorem, representation theorems, monotone preferences

1 Introduction

> "We would like to see it as a desideratum for normative decision theories that receipt of new information will always be appreciated." Wakker (1988)

Understanding the value of information has been in the minds of economists and statisticians for a long time. There is an older literature (e.g. Blackwell (1951), Marschak (1974), Gould (1974), and Allen (1983)), and a renewed interest in the value of information in recent years (e.g. Athey and Levin (1998), Persico (1996), and Persico (1999)).

In this paper we make four contributions to this literature. First we prove two impossibility theorems. We consider preference relations over information that are monotone, in the sense that more information is strictly preferred to less; we show that, if the state space is uncountable, no monotone preference relation over information can be represented by a utility function. That is, if a decision maker always prefers more information to less, his preferences over information structures cannot be represented by a utility function. The two theorems account for the two usual ways of modeling information: through partitions of the state space, and through σ-algebras.

Our result is important because it shows that utility theory is not likely to be a useful tool in the analysis of the value of information. This finding should be contrasted with the existing literature on the value of information, where utility representations are used. The use of a utility implies that preferences are not monotone. Besides making a contribution to the literature on the value of information, this result is also relevant for the literature on utility theory. In particular, economists have long studied the behavioral consequences of the existence of utility functions. For example, Koopmans (1960) showed that if a utility function for the uncountable set of infinite paths of consumption exists, the decision maker must exhibit impatience. Our result shows that if a utility function for information structures on an uncountable state space exists, the decision maker must exhibit indifference to information.[1]

Our second contribution is didactic. We give a simple proof of one of our impossibility theorems when the state space is [0,1]. We believe that this is a better example of non-representability than the usual textbook example, lexicographic preferences. Lexicographic preferences are not present in many economic applications, while problems involving the value of information are common. Our method of proof is essentially the same as that of the standard textbook proof of non-representability of lexicographic preferences.

Our third contribution is to show that monotone preferences over information are the first economic example of non-representability that is essentially different from lexicographic preferences. Recently, Beardon, Candeal, Herden, Induráin and Mehta (2000) have shown that there are exactly four classes of non-representable preferences, one of which is the set of preferences that are isomorphic to lexicographic preferences. Beardon et al. (2000) argue that all economic examples of non-representability belong to the lexicographic class; we show that monotone preferences over information belong to one of the other three classes (concretely, it is a long line, see below for a definition).

[1] We thank Stephen Morris for bringing this close connection to our attention.

Our final contribution is to show that monotone preferences over information on an uncountable state space are a violation of expected utility. Wakker (1988) suggests that (weakly) monotone preferences over information might imply expected utility. In light of our results, this is false. An expected utility maximizer's preferences induce an (indirect) utility function for information structures, and our main result shows that if preferences are monotone (and thus weakly monotone) they are not representable by a utility function.

1.1 Monotone preferences

The maintained assumption in the paper is that preferences are complete, transitive and monotone. Monotonicity in our context means that if partition (or σ-algebra) A is finer than partition (or σ-algebra) B, the decision maker strictly prefers A.

An objection to this assumption is that a decision maker (DM) who conforms to Savage's axioms, and thus has priors over the state space, will not have monotone preferences over information if the state space is uncountable. To see this, suppose that the state space is the interval $[0,1]$, and that DM's priors are represented by the uniform distribution. Then DM is indifferent between total ignorance and receiving a signal that tells her if the state $1/2$ has occurred or not. Ex-post knowledge of the state $1/2$ may be valuable, but since it is a probability zero event the signal is worthless to DM.

The source of the problem is not simply that priors rule out a large number of atoms. There are models of non-expected utility (e.g. Schmeidler (1989), Gilboa and Schmeidler (1989)) that allow an uncountable number of atoms. [2] Our result implies that, even for these models, a representation is impossible.

Still, we believe that monotonicity is a natural assumption for at least four reasons. First, it is dubious that, if asked, many people would be *exactly* indifferent between ignorance and the $1/2$-signal above. It is, after all, an empirical question: what is the best behavioral assumption for the analysis of information, Savage's axioms or monotonicity? The stage is indeed set for a "paradox," if people make monotone choices over information they cannot have priors.

The question then arises: how would one test for monotonicity?[3] We now turn to this issue, by describing a choice problem where the decision maker must first choose the information structure that he finds more useful for a second choice problem involving bets. Suppose that partition τ is finer than τ', so there is an element k' of τ' that is the union of a collection $\{k_\alpha\}$ of elements of τ. Let k be any element of the collection $\{k_\alpha\}$ and k^c the union of the rest of the elements of the collection. The individual must first choose between τ and τ'. Then, after he is informed in what element of the chosen partition the true state lies, he must choose between the following acts (bets)

$$f = \begin{cases} \text{z} & \text{if the state is in } k \\ 0 & \text{otherwise} \end{cases} \quad \text{and } f^c = \begin{cases} \text{z} & \text{if the state is in } k^c \\ 0 & \text{otherwise,} \end{cases}$$

[2] For example, a capacity can assign positive mass to an uncountable number of singletons. See section 2.4 for an example with maxmin preferences.

[3] We thank a referee for raising the issue of how to elicit preferences over information structures.

where z is a large sum of money. This is basically the $1/2$-signal example above. Our experience in classroom and seminar "experiments" is that an important proportion of people trained in probability and Bayesian decision-making, like economists, display monotone preferences in tests like this. For this reason we believe that many people do have monotone preferences.

A second reason why one should study the consequences of monotonicity was beautifully stated by Wakker (1988) in the quotation at the beginning: monotonicity is normatively a natural assumption, and therefore, its consequences must be investigated.

A third reason why monotonicity is relevant, is that the problem of whether an individual likes finer partitions is independent of, and may be more basic than, whether DM's preferences accord with Savage's theory. We may wish to analyze the robustness of a utility representation, in which case we need to analyze arbitrary preferences over information, and representation breaks down. In fact, representation rests on a large number of indifferences; any psychological wrinkle that could tilt this indifferences towards monotonicity makes any utility representation break down. In a vague sense, then, representable preferences over information are non-generic. To illustrate this point, we show how monotone preferences arise naturally if the individual is a maxminimizer. Since it has been argued that this may happen if DM is uncertainty averse, the experimental evidence that individuals dislike uncertainty suggests that monotone preferences may be empirically important.[4]

A fourth reason why we think that monotone preferences are important is that introspection and a very wide body of psychological research suggest that information has intrinsic value.[5] That is, people value information not only to make contingent plans, but also for its own sake. Psychologists have long recognized the importance of anticipatory feelings related to the acquisition of information and resolution of uncertainty. For instance, anxiety theory is today one of the most active areas of research in psychology. Of course, the desire to reduce anxiety will induce monotone preferences for information.[6] Grant et al. (1998) quote a physician as saying, about tests of incurable genetic disorders, that "There are some people who, even in the absence of being able to alter outcomes, find information of this sort beneficial." In those cases, even if people have priors over the state space, preferences for information will typically be monotone.

Finally, a comment on the criticism that monotone preferences are uninteresting because they preclude expected utility is in order. Taken seriously, this view implies that we should not study *any* problems beyond the realm of expected utility. Then, a host of interesting questions such as, just to name an example, the relation between risk and information, could not be analyzed.[7]

[4]See Gilboa and Schmeidler (1989) where the relation between uncertainty aversion and maxmin preferences is discussed.

[5]See, for example, Grant et al. (1998), Chew and Ho (1994) Ahlbrecht and Weber (1996) and the references cited therein.

[6]On the topic of anxiety and anticipatory feelings in economics, see Caplin and Leahy (2001).

[7]See Grant et al. (1998), Schlee (1990), Schlee (1991) Machina (1989) and Safra and Sulganik (1995) for more on this topic.

2 The Non-representation Theorems

One strategy for modeling information is to identify information with partitions of the state space; in this model, more information signifies a finer partition. In mathematics, this approach was initiated by Hintikka (1962), and introduced to economics by Aumann (1974). A preference relation on the set of all partitions is monotone if finer partitions are preferred to coarser partitions. The second approach is to model an agent's information by a σ-algebra over the state space—this approach is common in statistics, but also in economics and finance. A σ-algebra represents more information than another σ-algebra if it is finer. Preferences on the set of σ-algebras are monotone if, whenever one σ-algebra is contained in another, the larger one is preferred.

In this section we prove the main results of this paper: that monotone preferences over information can not be represented by a utility function if the state space is uncountable. In the next subsection we prove the result for the partitions approach, in Theorem 1. In the following subsection we prove it for the σ-algebra approach, in Theorem 2.

Theorems 1 and 2 are independent results, as the two approaches to modeling information are not equivalent, and neither model is more general than the other (see Dubra and Echenique (2000)).

2.1 Partitions

In this section we model information by partitions of a set of possible states of nature, Ω. A ***partition*** τ of Ω is a collection of pairwise disjoint subsets whose union is Ω; note that for each state of nature ω there is a unique element of τ that contains ω. A decision maker whose information is represented by τ is informed only that the element of τ that contains the true state of nature has occurred. In other words, the decision maker cannot distinguish between states that belong to the same element of τ.

A ***preference relation*** on a set X is a complete (total), transitive binary relation on X. Throughout this note, the symbol $\preceq$ will stand for a preference relation. A preference relation $\preceq$ is ***representable*** if there is a function $u : X \to \mathbf{R}$ such that $x \preceq y$ if and only if $u(x) \leq u(y)$.

Let $\mathcal{P}(\Omega)$ be the set of all partitions of Ω. If $\tau, \tau' \in \mathcal{P}(\Omega)$, say that τ' is ***finer*** than $\tau \neq \tau'$ if, for every $A \in \tau'$, there is B in τ such that $A \subseteq B$. A preference relation $\preceq$ on $\mathcal{P}(\Omega)$ is ***monotone*** if $\tau \prec \tau'$ whenever τ' is finer than τ.

Monotonicity is a natural assumption on preferences: if τ' is a finer partition than τ, then τ' contains more information.[8] The intuition is the following. Suppose a decision maker has information represented by τ'. When state ω occurs, she is informed of the event $B \in \tau'$. That is, she knows that some state in B has happened, but does not know which one exactly. If her information had been represented by τ, she would have known that a certain event $A \supseteq B$ occurred. In this case, she could not rule out states in A but not in B, whereas, if her partition is τ', she would know that states in $A \backslash B$ did not occur.

[8]Which does not contradict that τ' could have more information than τ and not be finer, only that refinement is sufficient for more information. So, our definition does not contradict the analysis in Athey and Levin (1998)

We now state our main theorem. It establishes that when the state space is uncountable, preferences that prefer more information to less cannot be represented by a utility function.

Theorem 1 *Let Ω be uncountable. If $\preceq$ on $\mathcal{P}(\Omega)$ is monotone then it is not representable.*

Remark. Although all the theorems in this note are stated for complete preorders, the proofs show that the theorems hold for possibly incomplete preorders. For example, Theorem 1 would say that if an incomplete preference relation is monotone, there does not exist a representable proper extension.[9]

All proofs, except that of Proposition 3, are presented in the appendix. To gain some intuition for why the theorem is true, recall that the representation of a preference relation is always a matter of how large indifference curves are—at one extreme, if an agent is indifferent between all elements of her choice set, then her preferences are represented by any constant function. Here, monotonicity of preferences over a large set, the set of partitions of an uncountable set, implies the existence of "too many" indifference curves. The proof of Proposition 3, in turn, gives a more precise intuition for why Theorem 1 is true.

2.2 σ algebras

In statistics and finance, but also in economics (see for example Allen (1983)), the information possesed by an individual is often modeled through a σ-algebra, and not a partition, on the space of states of nature. In this model, there is a primitive measurable space (Ω, F), and information is identified with sub-σ-algebras of F.

Let $(\Omega, 2^{\Omega})$ be the primitive measurable space. Let $\mathcal{F}(\Omega)$ be the set of all σ-algebras on Ω. If $F, G \in \mathcal{F}(\Omega)$, say that F is ***finer*** than G if G is a proper subset of F, noted $G \subsetneq F$. The intuition behind the use of σ-algebras is that if $A, B \subseteq \Omega$ are not measurable but $A \cup B$ is, then the decision maker cannot distinguish between states in A and states in B; she can distinguish between states in $A \cup B$ and in $(A \cup B)^c$. Thus if F is finer than G, then F represents more information than G.

A preference relation $\preceq$ on $\mathcal{F}(\Omega)$ is ***monotone*** if $G \prec F$ whenever F is finer than G.

Theorem 2 *Let Ω be uncountable. If $\preceq$ on $\mathcal{F}(\Omega)$ is monotone then it is not representable.*

2.3 Theorem 1 junior grade

Theorems 1 and 2 show that utility theory is not a useful tool in the analysis of the value of information. Besides this substantive contribution, we can also make a didactic contribution by providing a simple example of non-representability. The canonical example of non-representability is lexicographic preferences, but the only place students of economics find lexicographic preferences is in discussions of representability. We believe that preferences over information is a more relevant example of non-representability. Proposition 3 shows that

[9] A preorder $\leq$ is a proper extension of $\preceq$ if $p \prec q$ implies $p < q$. See Dubra and Ok (2000).

no monotone preference over information on partitions of $[0,1]$ is representable. The method of proof is basically the same as for lexicographic preferences.

Proposition 3 *Let $\Omega = [0,1]$. If $\preceq$ on $\mathcal{P}(\Omega)$ is monotone then it is not representable.*

Proof. Suppose, by way of contradiction, that there is a function $u : \mathcal{P}(\Omega) \to \mathbf{R}$ that represents $\preceq$. For each $x \in (0,1)$ let

$$\tau_x = \{\{y\} : 0 \leq y < x\} \cup [x,1],$$
$$\tau'_x = \{\{y\} : 0 \leq y \leq x\} \cup (x,1].$$

Note that $\tau_x, \tau'_x \in \mathcal{P}(\Omega)$, and that $\tau_x \prec \tau'_x$, as τ'_x is finer than τ_x. But then there is a rational number $r(x)$ such that $u(\tau_x) < r(x) < u(\tau'_x)$. Let $x \neq \tilde{x}$, say $x < \tilde{x}$, then $\tau_{\tilde{x}}$ is finer than τ'_x. Thus

$$u(\tau_x) < r(x) < u(\tau'_x) < u(\tau_{\tilde{x}}) < r(\tilde{x}) < u(\tau'_{\tilde{x}}).$$

But then r is injective, a contradiction. ■

Remark. Non-representability in general uncountable subsets of $\mathbf{R}$ can be proven by a slight modification of the proof of proposition 3.

2.4 An example: Maxmin Preferences

An expected-utility-maximizer does not have monotone preferences over information. Here we present an example of a decision problem with maxmin preferences, under our assumptions, the derived value of information is such that being informed in a particular state makes DM always strictly better off. Because of this monotonicity, her preferences are not representable by a utility.

Let $\Omega = [0,1]$ and P a set of probability measures on Ω. DM must choose an element (action) in $A = [0,1]$ after observing a signal about the state of nature. Her state-contingent utility is given by $u(\omega, a) = -(\omega - a)^2$ (e.g. DM is a statistician seeking to minimize the mean squared error). We will assume that DM is a maxminimizer, so the utility in event B when action a is chosen is

$$U(B,a) = \inf_{p(B)>0} \int_B \frac{u(\tilde{\omega}, a)}{p(B)} dp(\tilde{\omega}).$$

We need $\max U(B,a)$ to be well defined, so that $a[B]$, the optimal action in event B exists. For example, if P contains all degenerate priors on Ω, then $\max U(B,a)$ is well defined. To see this, let $\overline{B}$ stand for the closure of B, and $a_{B,a} \in \arg\min_{\omega \in \overline{B}} (\omega - a)^2$, we have that $U(B,a) = u(a_B, a)$. Therefore, $U(B,a)$ is a continuous function of a, and $a[B]$, the optimal action in event B is well defined.

A set P of probability measures over Ω is ***broad*** if the set of $\omega \in \Omega$ such that $p(\omega) > 0$ for some $p \in P$ is uncountable. This is the case, for example, if P contains all degenerate probability measures. Natural choices of P are broad, for example the set of all priors, or, given a prior p, the "ε-contaminated" set of all $\varepsilon p + (1-\varepsilon)p'$ for arbitrary p'. [10]

[10]This remark is due to an anonymous referee.

We will assume that DM's preferences over partitions satisfy the following axiom.

Dominance. If for all $\omega \in \Omega$,

$$U\left(k_{\tau}(\omega) \cap k_{\tau'}(\omega), a\left[k_{\tau}(\omega)\right]\right) \geq U\left(k_{\tau}(\omega) \cap k_{\tau'}(\omega), a\left[k_{\tau'}(\omega)\right]\right)$$

and there exists $\tilde{\omega}$ and $p \in P$ with $p(\tilde{\omega}) > 0$ such that the above inequality is strict, then $\tau \succ \tau'$.

DM is comparing two partitions τ and τ'. In doing so, she imagines herself in a fixed event $k_{\tau}(\omega) \cap k_{\tau'}(\omega)$. Suppose she realizes that the utility she would obtain by choosing the τ-optimal action in any of the states in that event is weakly larger than that she would obtain from the τ'-optimal action. Suppose in addition that DM believes that, with positive probability, a state will occur in which, choosing the optimal action under τ will make her strictly better off than choosing the optimal action under τ'. Then she should strictly prefer partition τ over τ'. We shall assume that DM uses Bayesian updating on all priors in P, this is for simplicity, there are other choices (Gilboa and Schmeidler, 1993).

Proposition 4 *Let $\preceq$ be a preference relation over $\mathcal{P}(\Omega)$. If P is broad, $U(B,a)$ is continuous for all B, and $\preceq$ satisfies dominance, then $\preceq$ is not representable.*

Remark. If dominance is strengthened so that the conclusion follows without requiring $p(\omega) > 0$, then we obtain non-representation also for expected utility. We use maxmin as a natural way of incorporating multiple priors, and thus an uncountable number of atoms.

3 Recovering the representation.

Given the negative result in Theorem 1, one may wonder under what conditions one can recover a utility representation for preferences for information. In this section we discuss the existence of a representation when Ω is countable, and then present two alternative models that yield a representation, and comment on their relative merits.

In what follows we will only deal with the partitions model because we believe that this is the more natural way to model information.

3.1 Countable Ω

It is natural to ask if Theorem 1 can be strengthened to countable Ω. Example 5 shows that it can not. When Ω is countable, there are monotone preferences over information that are representable. Example 5 may be somewhat misleading, though. We show (Theorem 6) that, if preferences are monotone, but individual states are still relatively unimportant, then there is a utility if and only if Ω is finite.

Example 5 *Consider $\Omega = \left\{(1/2)^i : i \in \mathbf{N}\right\}$. Any $\tau \in \mathcal{P}(\Omega)$ has at most a countable number of elements, say $\tau = \{A_k : k \in \mathbf{N}\}$ (if τ has a finite number of elements, put $A_k = \emptyset$ as often as necessary). Let*

$$u(\tau) = \sum_{k \in \mathbf{N}, A_k \neq \emptyset} \inf A_k.$$

Then u represents a monotone preference relation over information on Ω (namely the preference relation induced by u).

Let Ω be a set and $\preceq$ a preference relation on $\mathcal{P}(\Omega)$. An element $\omega \in \Omega$ is an ***atom*** for $\preceq$ if, for any $A \subseteq \Omega$ with $\omega \in A$ and at least two elements,

$$\{A, A^c\} \preceq \tau \preceq \{\{\omega\}, A \setminus \{\omega\}, A^c\}$$

is satisfied only for $\tau = \{A, A^c\}$ or $\tau = \{\{\omega\}, A \setminus \{\omega\}, A^c\}$. A state of nature is an atom if the decision maker gains relatively little from being perfectly informed about this state—in the sense that any partition that is preferred over $\{A, A^c\}$ is also preferred over $\{\{\omega\}, A \setminus \{\omega\}, A^c\}$.

Theorem 6 *Let Ω be a set. A monotone preference relation on $\mathcal{P}(\Omega)$ that has an atom is representable if and only if Ω is finite.*

3.2 Priors on Ω and worthless states.

We argued in the Introduction that the existence of priors on the set of states of nature could imply that preferences are not monotone. We present a simple model where a utility representation for partitions arises. Versions of this model are used in many papers on the value of information (e.g. Blackwell (1951) and Athey and Levin (1998)).

We shall now rule out intrinsic preferences for information, and only consider preferences for information derived from the role of information in guiding choices.

There is a set Ω of states of nature. DM must choose an action, an element in a compact set A, after observing a signal about the state of nature. DM's prior knowledge is represented by the probability measure μ over Ω, given a probability space (Ω, F, μ). In this section, $\mathcal{P}(\Omega)$ will stand for the set of measurable partitions. Let $u : \Omega \times A \to \mathbf{R}$ be DM's (measurable) state-contingent utility function. Given any partition $\tau \in \mathcal{P}(\Omega)$ and $\omega \in \Omega$, let $k_\tau(\omega)$ be the element of τ that contains ω. When ω is realized, the decision maker is informed that an element in $k_\tau(\omega)$ has occurred. Let

$$a^*(\omega) \in \arg\max_{a \in A} \int_{k_\tau(\omega)} u(\tilde{\omega}, a) d\mu(\tilde{\omega}),$$

so that for each ω, $a^*(\omega)$ is DM's optimal choice, given her signal $k_\tau(\omega)$ (in fact the selection $a(.)$ can be taken to be measurable). We say that a preference relation $\preceq$ on $\mathcal{P}(\Omega)$ is ***derived from priors*** if it is represented by a utility function $U : \mathcal{P}(\Omega) \to \mathbf{R}$ such that

$$U(\tau) = \int_\Omega \left\{ \int_{k_\tau(\omega)} u(\tilde{\omega}, a^*(\omega)) d\mu(\tilde{\omega}) \right\} d\mu(\omega)$$

for some action space A, $u : \Omega \times A \to \mathbf{R}$ and beliefs μ.

As a trivial corollary of Theorem 1, we obtain the following result.

Corollary 7 *If $\preceq$ on $\mathcal{P}(\Omega)$ is monotone and Ω is uncountable, then $\preceq$ is not derived from priors.*

To see why the resulting preference relation is not monotone, let all singleton sets be measurable (i.e. $\{\omega\} \in F$ for all $\omega \in \Omega$) and note that all but a countable number of ω have zero probability. Then, since it is worthless to be perfectly informed in a zero probability event, DM's utility is not higher after a refinement of a zero probability ω. Thus, requiring that DM has priors is like reducing the size of Ω.

Note that the construction of U requires a good deal of faith in the setup. If we wish to analyze the robustness of the U construction we would need to consider preferences over $\mathcal{P}(\Omega)$, and representation is no longer guaranteed.

3.3 Finite Action Space.

The value of more information, of a finer partition, is that DM has less restrictions on her choice of action. DM must choose the same action at states that she cannot distinguish between, so a finer partition eases some restrictions and thus must make DM (weakly) better off. If DM faces a limited number of alternative actions, more information may not always make a difference—DM will not strictly gain from more information. Thus, a limit on the number of possible choices has much the same effect as the existence of priors, it limits the value of being informed in particular states.

The setup in this sub-section is the same as in 3.2, only we now allow for more general preferences. The set of states of nature is Ω, DM must choose an action in A after observing a signal about the state of nature. The primitives of the model are a collection $\{\preceq_B\}_{B \in 2^\Omega}$ of preference relations over A, and a preference relation $\preceq$ over $\mathcal{P}(\Omega)$.

The interpretation of $\preceq_B$ for a fixed subset B of Ω is the following. Suppose state ω occurs and DM is informed of the element of the partition that has occurred, say $B = k_\tau(\omega)$. Given this, she chooses an action that is maximal according to $\preceq_B$. Say that $a_\tau(\omega)$ is the maximal action according to $\preceq_{k_\tau(\omega)}$. Thus, each partition τ generates a function $a_\tau : \Omega \to A$. Let $f : \mathcal{P}(\Omega) \to A^\Omega$ be the map that takes partitions into functions from Ω to A : $f(\tau) = a_\tau$.

DM is also endowed with the preference relation $\preceq$ on $\mathcal{P}(\Omega)$, which is assumed to be consistent with the collection $\{\preceq_B\}$ in the sense that, if two partitions τ and τ' are such that $a_\tau = a_{\tau'}$, $\tau \sim \tau'$.

The next proposition shows, as was argued in the beginning of this section, that reducing the number of actions DM can adopt enables representation of her preferences.

Proposition 8 *If Ω is a compact metric space, A is finite, and $f(\tau) \in A^\Omega$ is continuous for all τ, then $\preceq$ is representable.*

4 Preferences over information are not lexicographic

> "So the answer to the crucial question in utility theory about whether or not the only non-representable preference relation is essentially the Debreu (lexicographic) chain is, somewhat informally, yes provided that we do not want examples based on ordinal numbers with large cardinality." Beardon et al. (2000)

Theorem 9 below shows that a monotone preference over an uncountable state space is essentially different from lexicographic preferences. As was shown in Proposition 3 one can build an example of a non representable preference relation that makes no explicit use of ordinal numbers. Still, of course, the reason why representability fails is the large cardinality of the set of all partitions on Ω : non-representability in Theorem 1 comes from the existence of too many partitions to be ranked strictly. The existence of a utility would imply that there are "only" a continuum many partitions that can be strictly ranked.

The ***dual*** order of a given ordered set $\langle X, \preceq \rangle$ is the order $\preceq_d$ on X defined by $x \prec_d y$ if and only if $y \prec x$. Let γ be the first uncountable ordinal. An ordered set $\langle X, \preceq \rangle$ is ***long*** if $\langle X, \preceq \rangle$, or $\langle X, \preceq_d \rangle$, contain a sub-chain which is order-isomorphic to $[0, \gamma)$. [11]

Theorem 9 *Let Ω be uncountable. If $\preceq$ on $\mathcal{P}(\Omega)$ is monotone then $\langle \mathcal{P}(\Omega), \preceq \rangle$ is long.*

Beardon et al. (2000) show that, if $\langle \mathcal{P}(\Omega), \preceq \rangle$ is long, it is not order-isomophic to the lexicographic line. So, our non-representation theorem is essentially different from the lexicographic result.

5 Concluding Remarks

In large sets, the representation of a decision maker's preferences by a utility depends on the "size" of her indifference curves. At one extreme, if DM is indifferent between all possible states her preferences are trivially representable; this is also the case if DM has a finite number of indifference curves. Preferences over information are typically weakly monotone, in the sense that more information is weakly preferred to less. We show that if indifference is ruled out for a large enough set of states by requiring strict monotonicity, there is no utility representation for preferences over information.

The question of weak vs. strict monotonicity is reminiscent of preferences over sequences of outcomes in repeated games. The "overtaking criterion" assumes that no outcome in an individual time period is important, while the "discounting criterion" assumes that a change in payoffs in any single time period makes a difference. Here, as in repeated games, both assumptions have their merit. But, unlike in repeated games, here they give very different conclusions. When individual states are unimportant (e.g. because of Savage's axioms, or because there are few alternative actions) there is a utility, but when enough states are important there is none. In our opinion this implies that any representation of preferences over information is not robust to changes in the environment.

[11] See Beardon et al. (2000) for details.

6 Appendix

Proof of Theorem 1. Let $\leq$ linearly order Ω (such an order exists, for example, let $\leq$ well order Ω). For all $\omega \in \Omega$, define $\tau_\omega, \tau'_\omega \in \mathcal{P}(\Omega)$ by

$$\tau_\omega = \{\{\theta\} : 0 \leq \theta < \omega\} \cup \{\theta : \omega \leq \theta\},$$
$$\tau'_\omega = \{\{\theta\} : 0 \leq \theta \leq \omega\} \cup \{\theta : \omega < \theta\}.$$

Note that τ'_ω is finer than τ_ω, and that if $\omega < \hat{\omega}$, then $\tau'_\omega \preceq \tau_{\hat{\omega}}$.

Suppose, by way of contradiction, that there is a utility $u : \mathcal{P}(\Omega) \to \mathbf{R}$ that represents $\preceq$. Then, for each $\omega \in \Omega$ there is a rational number $r(\omega)$ such that $u(\tau_\omega) < r(\omega) < u(\tau'_\omega)$. Let $\omega \neq \hat{\omega}$, say $\omega < \hat{\omega}$, then $r(\omega) < u(\tau'_\omega) \leq u(\tau_{\hat{\omega}}) < r(\hat{\omega})$. Thus $r : \Omega \to \mathbf{Q}$ is an injection, a contradiction as Ω is uncountable. ■

Proof of Theorem 2. Let $\leq$ linearly order Ω, and endow Ω with the order-interval topology. For all $\omega \in \Omega$, let $\mathcal{B}_\omega$ denote the Borel σ-algebra on $\{\theta : \theta < \omega\}$, and $\mathcal{B}'_\omega$ the Borel σ-algebra on $\{\theta : \theta \leq \omega\}$. To each ω we associate two σ algebras σ_ω and σ'_ω defined by

$$\begin{aligned} \sigma_\omega &= \mathcal{B}_\omega \cup \{B \cup \{\theta : \omega \leq \theta\} : B \in \mathcal{B}_\omega\} \\ \sigma'_\omega &= \mathcal{B}'_\omega \cup \left\{B \cup \{\theta : \omega < \theta\} : B \in \mathcal{B}'_\omega\right\} \end{aligned}$$

First, it is easy to check that σ_ω and σ'_ω are indeed σ algebras. Second, $\sigma_\omega \subseteq \sigma'_\omega$, as any $\{\theta : \theta < \omega\}$-open set is open and contained in $\{\theta : \theta \leq \omega\}$. Then, $\{\omega\} \in \sigma'_\omega$ and $\{\omega\} \notin \sigma_\omega$ imply that $\sigma_\omega \subsetneq \sigma'_\omega$.

Suppose, by way of contradiction, that there is a utility $u : \mathcal{F}(\Omega) \to \mathbf{R}$ that represents $\preceq$. Monotonicity ensures that one can assign to each ω a rational $r(\omega)$ such that

$$u(\sigma_\omega) < r(\omega) < u(\sigma'_\omega).$$

Now pick any $\beta \in \Omega$, say $\omega < \beta$. Since any $\{\theta : \theta \leq \omega\}$-closed set is $\{\theta : \theta < \beta\}$-closed, $\mathcal{B}'_\omega \subseteq \mathcal{B}_\beta$. Then, $\{\theta : \omega < \theta \leq \beta\} \in \mathcal{B}_\beta$ implies that $\sigma'_\omega \subseteq \sigma_\beta$. Thus,

$$u(\sigma_\omega) < r(\omega) < u(\sigma'_\omega) \leq u(\sigma_\beta) < r(\beta) < u(\sigma'_\beta),$$

and r is injective, a contradiction. ■

Proof of Proposition 4. We now show that, if a partition τ is a "one-point refinement" of τ', then $\tau \succ \tau'$. Pick any $k \in \tau'$ with at least two elements, and fix $\omega \in k$ with $\omega \neq a[k]$. We will now show that

$$\tau = \left\{l \cap \{\omega\} : l \in \tau'\right\} \cup \left\{l \cap \{\omega\}^c : l \in \tau'\right\}$$

and τ' satisfy dominance (τ is a one-point refinement of τ').

Note that for all $\omega' \notin k_{\tau'}(\omega)$, we have that $k_{\tau'}(\omega') = k_\tau(\omega')$ and thus

$$U\left(k_\tau(\omega') \cap k_{\tau'}(\omega'), a\left[k_\tau(\omega')\right]\right) = U\left(k_\tau(\omega') \cap k_{\tau'}(\omega'), a\left[k_{\tau'}(\omega')\right]\right)$$

Now, fix any $\omega' \in k$. Two cases must be considered.
I) $\omega' = \omega$. In this case, $k_\tau(\omega') \cap k_{\tau'}(\omega') = \{\omega\}$. Since P is broad there is $p \in P$ with $p(\{\omega\}) > 0$. Since $\omega \neq a[k]$ we have

$$\begin{aligned} U(\{\omega\}, a[k_{\tau'}(\omega)]) &= U(\{\omega\}, a[k]) \\ &\leq \int_{\{\omega\}} \frac{u(\tilde{\omega}, a[k])}{p(\{\omega\})} dp(\tilde{\omega}) \\ &= u(\{\omega\}, a[k]) \\ &< u(\omega, \omega) = U(\{\omega\}, a[k_\tau(\omega)]) \end{aligned}$$

II) $\omega' \neq \omega$. In this case, $k_\tau(\omega') \cap k_{\tau'}(\omega') = k_\tau(\omega')$. Then, by definition $U(k_\tau(\omega'), a[k_\tau(\omega')]) \geq U(k_\tau(\omega'), a[k_{\tau'}(\omega')])$.

In fact, monotonicity to one-point refinements is all that is needed in the proof of Theorem 1. Thus $\preceq$ is not representable. ■

Proof of Theorem 6. The proof makes use of the classical representation theorem of Garrett Birkhoff (see Theorem 3.5 in Kreps (1988)): a preference relation $\preceq$ on a choice space X is representable if and only if X is order-separable; that is, if and only if there is $Z \subseteq X$, countable, such that $x, y \in X$, $x \prec y$ imply that there is $z \in Z$ with $x \preceq z \preceq y$.

(if) If Ω is finite, then $\mathcal{P}(\Omega)$ is finite and therefore order-separable. By Birkoff's Theorem, $\preceq$ is representable.

(only if) Let Ω be infinite and $\omega \in \Omega$ an atom for $\preceq$. There is an uncountable number of sets A that contain ω and have at least another element. Let $p(A) = \{A, A^c\}$ and $p'(A) = \{\{\omega\}, A \setminus \{\omega\}, A^c\}$. Note that $p(A), p'(A) \in \mathcal{P}(\Omega)$ and that $p(A) \prec p'(A)$. Also note that there is no $x \in \mathcal{P}(\Omega)$ with $p(A) \prec x \prec p'(A)$. Order-separability would require that there be $z \in Z$ with $p(A) \preceq z \preceq p'(A)$ i.e. that either $p(A)$ or $p'(A)$ be in Z. Since Ω is not finite, it has uncountably many subsets like A, hence Z could not be countable. By Birkhoff's theorem, there is no utility representation. ■

Proof of Theorem 9. For any ordered set $\langle X, \preceq \rangle$, let $(x, y) \equiv \{z \in X : x \prec z \prec y\}$, for x, y in X.

Let $\leq$ well-order Ω. We shall construct an uncountable collection of intervals in $\mathcal{P}(\Omega)$. Let

$$\begin{aligned} \tau_\omega &= \{\{\theta\} : \theta < \omega\} \cup \{\theta : \omega \leq \theta\}, \\ \tau &= \{\{\omega\} : \omega \in \Omega\}. \end{aligned}$$

Since $\preceq$ is monotonic, for all $\omega < \theta$, $\tau_\omega \prec \tau_\theta$. The collection of intervals $\{(\tau_\omega, \tau)\}_{\omega \in \Omega}$ is well ordered by set inclusion, as Ω is well ordered. Theorem 3.1 of Beardon et al. (2000) then ensures that $\langle \mathcal{P}(\Omega), \preceq \rangle$ is long. ■

Proof of Proposition 8. Let $C(\Omega, A)$ denote the space of continuous functions from Ω to A, endowed with the topology of uniform convergence. If A is finite and Ω a compact metric space, $C(\Omega,A)$ is a separable metric space (see Aliprantis and Border (1999, Theorem 3.85)). Since separability is hereditary in metric spaces, $f(\mathcal{P}(\Omega))$ is a separable metric space. Thus, by Debreu (1954, Theorem II) any continuous preference relation on $f(\mathcal{P}(\Omega))$ is representable.

Let the preference relation $\trianglelefteq$ on $f(\mathcal{P}(\Omega))$ be defined by $a_\tau \trianglelefteq a_{\tau'}$ if and only if $\tau \preceq \tau'$. A convergent sequence in $C(\Omega,A)$ is eventually constant, as A is finite and $C(\Omega,A)$ is endowed with the topology of uniform convergence. Thus $\trianglelefteq$ is continuous. By Debreu (1954, Theorem II) there is a utility function $u : f(\mathcal{P}(\Omega)) \to \mathbf{R}$ that represents $\trianglelefteq$. Defining $v : \mathcal{P}(\Omega) \to \mathbf{R}$ by $v(\tau) = u(f(\tau))$ we see that v represents $\preceq$. ■

References

Ahlbrecht, Martin and Martin Weber, “The Resolution of Uncertainty: An Experimental Study,” *Journal of Institutional and Theoretical Economics*, December 1996, *152* (4), 593–607.

Aliprantis, Charalambos D. and Kim C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, 1999.

Allen, Beth, “Neighboring Information and Distributions of Agents’ Characteristics Under Uncertainty,” *Journal of Mathematical Economics*, 1983, *12*, 63–101.

Athey, Susan and Jonathan Levin, “The Value of Information in Monotone Decision Problems,” November 1998. Mimeo M.I.T.

Aumann, Robert J., “Subjectivity and Correlation in Randomized Strategies,” *Journal of Mathematical Economics*, 1974, pp. 67–96.

Beardon, Alan F., Juan C. Candeal, Gerhard Herden, Esteban Induráin, and Ghanshyam B. Mehta, “The Non-Existence of a Utility Function and the Structure of Non-Representable Preference Relations,” 2000. Mimeo.

Blackwell, David, “Comparison of Experiments,” in J. Neyman, ed., *Proceedings of the second Berkeley symposium on mathematical statistics and probability*, University of California Press, Berkeley and Los Angeles 1951, pp. 93–102.

Caplin, Andrew and John Leahy, “Psychological Expected Utility Theory and Anticipatory Feelings,” *Quarterly Journal of Economics*, 2001, *116* (1), 55–79.

Chew, Soo Hong and Joanna L. Ho, “Hope: An Empirical Study of Attitude toward the Timing of Uncertainty Resolution,” *Journal of Risk and Uncertainty*, May 1994, *8* (3), 267–88.

Debreu, Gerard, "Representation of a Preference Ordering by a Numerical Function," in R.M. Thrall, C.H. Coombs, and R.L. Davis, eds., *Decision Processes*, Wiley, New York, 1954, pp. 159–65.

Dubra, Juan and Efe Ok, "A Model of Procedural Decision Making in the presence of Risk," forthcoming International Economic Review 2000.

—— **and Federico Echenique**, "Measurability is not About Information," Mimeo 2000.

Gilboa, Itzhak and David Schmeidler, "Maxmin Expected Utility with Non-unique Prior," *Journal of Mathematical Economics*, 1989, *18* (2), 141–53.

—— **and** —— , "Updating Ambiguous Beliefs," *Journal of Economic Theory*, 1993, *59* (1), 33–49.

Gould, J. P., "Risk, stochastic preference, and the value of information," *Journal of Economic Theory*, 1974, *8*, 64–84.

Grant, Simon, Atsushi Kajii, and Ben Polak, "Intrinsic Preference for Information," *Journal of Economic Theory*, 1998, *83* (2), 233–259.

Hintikka, Jakko, *Knowledge and Belief*, Cornell University Press, 1962.

Koopmans, Tjalling C., "Stationary Ordinal Utility and Impatience," *Econometrica*, April 1960, *28* (2), 287–309.

Kreps, David, *Notes on the theory of choice*, Westview Press, Colorado, 1988.

Machina, Mark J., "Dynamic Consistency and Non-expected Utility Models of Choice under Uncertainty," *Journal of Economic Literature*, December 1989, *27* (4), 1622–68.

Marschak, Jacob, *Economic Information, Decision and Prediction*, Dordrecht, D. Reidel. Theory and Decision Library, 1974.

Persico, Nicola, "Information Acquisition in Affiliated Decision Problems," feb 1996. Mimeo, UCLA.

—— , "Information Acquisition in Auctions," mar 1999. Forthcoming in Econometrica.

Safra, Zvi and Eyal Sulganik, "Schur Convexity, Quasi-convexity and Preference for Early Resolution of Uncertainty," *Theory and Decision*, September 1995, *39* (2), 213–18.

Schlee, Edward E., "Multivariate Risk Aversion and Consumer Choice," *International Economic Review*, August 1990, *31* (3), 737–45.

—— , "The Value of Perfect Information in Nonlinear Expected Utility Theory," *Theory and Decision*, 1991, *30*, 127–131.

Schmeidler, David, "Subjective Probability and Expected Utility without Additivity," *Econometrica*, May 1989, *57* (3), 571–587.

Wakker, Peter P., "Nonexpected Utility a Aversion of Information," *Journal of Behavioral Decision Making*, 1988, *1* (1), 169–175.

Colophon

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